Unit 6 Progress Check: FRQ Part A

1. NO CALCULATOR IS ALLOWED FOR THIS QUESTION.

Show all of your work, even though the question may not explicitly remind you to do so. Clearly label any functions, graphs, tables, or other objects that you use. Justifications require that you give mathematical reasons, and that you verify the needed conditions under which relevant theorems, properties, definitions, or tests are applied. Your work will be scored on the correctness and completeness of your methods as well as your answers. Answers without supporting work will usually not receive credit.

Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

t (hours)	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$R(t)$ (gallons per hour)	11	8	5	0

The rate at which water leaks from a container is modeled by the twice-differentiable function R, where $R(t)$ is measured in gallons per hour and t is measured in hours for $0 \le t \le 1$. Values of $R(t)$ are given in the table above for selected values of t.

(a) Use the data in the table to find an approximation for $R'\left(\frac{1}{2}\right)$. Show the computations that lead to your answer. Indicate units of measure.

Please respond on separate paper, following directions from your teacher.

(b) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate $\int_0^1 R(t)\,dt$. Indicate units of measure.

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Please respond on separate paper, following directions from your teacher.

(c) Use the data in the table to evaluate $\int_0^{\frac{1}{3}} R'(t)\,dt$.

Please respond on separate paper, following directions from your teacher.

(d) The sum $\sum_{k=1}^{n} R\left(\frac{1}{4}+\frac{k}{2n}\right)\frac{1}{2n}$ is a right Riemann sum with n subintervals of equal length. The limit of this sum as n goes to infinity can be interpreted as a definite integral. Express the limit as a definite integral.

Please respond on separate paper, following directions from your teacher.

Part A

Numerical answers do not need to be simplified for the approximation point. The approximation must include a substitution of the function values from the table into the difference quotient.

Select a point value to view scoring criteria, solutions, and/or examples to score the response.

0	1	2

The student response accurately includes both of the criteria below.

- ☐ approximation
- ☐ units

Solution:

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$R'\left(\frac{1}{2}\right) \approx \frac{R\left(\frac{2}{3}\right)-R\left(\frac{1}{3}\right)}{\frac{2}{3}-\frac{1}{3}} = \frac{5-8}{\frac{1}{3}} = -9$ gallons per hour per hour

Part B

To earn the first point, the response must present the form of a left Riemann sum. Therefore, $\frac{1}{3}(11+8+5)$ earns the first and second points.

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1	2	3 ✓

The student response accurately includes all three of the criteria below.

- ☐ left Riemann sum
- ☐ approximation
- ☐ units

Solution:

$$\int_0^1 R(t)\ dt \approx \frac{1}{3}\left(R(0)+R\left(\frac{1}{3}\right)+R\left(\frac{2}{3}\right)\right)$$
$$= \frac{1}{3}(11+8+5) = 8 \text{ gallons}$$

Part C

The response should include supporting work for the numerical answer of -3.

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1 ✓

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The student response accurately includes a correct value.

Solution:

$$\int_0^{\frac{1}{3}} R'(t)\ dt = R\left(\frac{1}{3}\right) - R(0) = 8 - 11 = -3$$

Part D

The first point may be earned as presented in the solution or by using that information to determine that the interval of integration is of length $\frac{1}{2}$. A response that produces the correct definite integral earns all 3 points.

Select a point value to view scoring criteria, solutions, and/or examples to score the response.

0	1	2	3

The student response accurately includes all three of the criteria below.

- ☐ $\Delta t = \frac{1}{2n}$
- ☐ limits of integration
- ☐ integrand

Solution:

For this Riemann sum, $\Delta t = \frac{1}{2n}$. Thus, the integral is over an interval of length $\frac{1}{2}$. This interval can be taken to be $\left[\frac{1}{4}, \frac{3}{4}\right]$.

The right endpoints of the subintervals used in the Riemann sum would be of the form $t_k = \frac{1}{4} + \frac{k}{2n}$ for k from 1 to n.

Therefore, $\lim\limits_{n\to\infty}\left(\sum_{k=1}^{n} R\left(\frac{1}{4} + \frac{k}{2n}\right) \cdot \frac{1}{2n}\right) = \int_{\frac{1}{4}}^{\frac{3}{4}} R(t)\ dt.$

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Note: The limit of the Riemann sum can be written as any definite integral of the form $\int_a^{a+\frac{1}{2}} R\left(\frac{1}{4} - a + t\right)\, dt.$

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2. NO CALCULATOR IS ALLOWED FOR THIS QUESTION.

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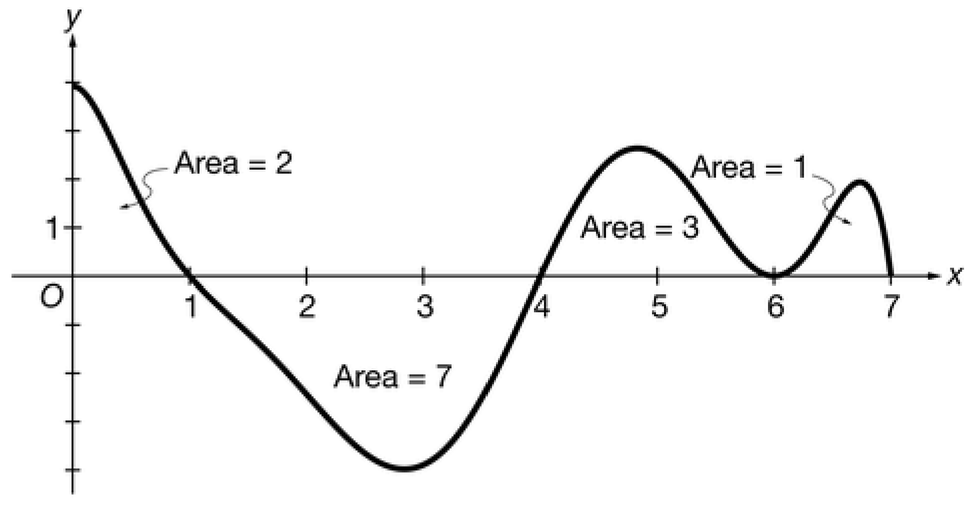

Graph of f'

The figure above shows the graph of f', the derivative of a differentiable function f, on the closed interval $0 \le x \le 7$. The areas of the regions between the graph of f' and the x-axis are labeled in the figure. The function f is defined for all real numbers and satisfies $f(4) = 10$.

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Let g be the function defined by $g(x) = 5 - x^2$.

(a) Find the value of $\int_0^7 f'(x)\,dx$.

Please respond on separate paper, following directions from your teacher.

(b) Given that $f(4) = 10$, write an expression for $f(x)$ that involves an integral. Use this expression to find the absolute minimum value of f and the absolute maximum value of f on the closed interval $0 \le x \le 7$. Justify your answers.

Please respond on separate paper, following directions from your teacher.

(c) Find $\int g(x)\,dx$.

Please respond on separate paper, following directions from your teacher.

(d) Find the value of $\int_1^2 x\,f'(g(x))\,dx$.

Please respond on separate paper, following directions from your teacher.

Part A

The response should include supporting work for the numerical answer of -1.

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1

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The student response accurately includes a correct value.

Solution:

$$\int_0^7 f'(x)\ dx = 2 - 7 + 3 + 1 = -1$$

Part B

The first point is earned for a correct expression for $f(x)$ OR evidence of correct use of FTC in determining the appropriate $f(x)$ values for consideration.

For the second point: The response may also consider $x = 6$, though this is not required in order to justify an absolute minimum value or an absolute maximum value on the interval.

For the third and fourth points: At most 1 out of 2 points is earned for supporting work that justifies a correct absolute minimum value OR absolute maximum value based on a Candidates Test with a maximum of one arithmetic error.

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1	2	3	4

The student response accurately includes all four of the criteria below.

- ☐ use of Fundamental Theorem of Calculus
- ☐ considers $x = 1$ and $x = 4$
- ☐ answers for absolute minimum value and absolute maximum value
- ☐ justification

Solution:

$$f(x) = f(4) + \int_4^x f'(t)\ dt$$

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The absolute minimum and absolute maximum will occur at a critical point where $f'(x) = 0$ or at an endpoint.

$f'(x) = 0 \;\Rightarrow\; x = 1,\; x = 4,\; x = 6$

The critical point at $x = 6$ is neither the location of an absolute minimum nor an absolute maximum because f' does not change sign at $x = 6$. Thus, the only candidates are $x = 0,\; x = 1,\; x = 4$, and $x = 7$.

x	$f(x)$
0	$f(4)+\int_4^0 f'(t)\,dt = f(4)-\int_0^4 f'(t)\,dt$ $=10-(2-7)=15$
1	$f(4)+\int_4^1 f'(t)\,dt = f(4)-\int_1^4 f'(t)\,dt$ $=10-(-7)=17$
4	10
7	$f(4)+\int_4^7 f'(t)\,dt = 10+(3+1)=14$

On the interval $0 \le x \le 7$, the absolute minimum value is $f(4) = 10$ and the absolute maximum value is $f(1) = 17$.

Part C

A response with any errors, including omission of the constant of integration, does not earn the point.

Select a point value to view scoring criteria, solutions, and/or examples to score the response.

0	1

The student response accurately includes a correct answer.

Solution:

$$\int g(x)\ dx = \int \left(5 - x^2\right)\ dx = 5x - \tfrac{1}{3}x^3 + C$$

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Part D

To earn the second point, a response must present a definite integral that correctly handles the substitution of the limits of integration. Correct handling of the reversal of the limits of integration and all calculations are part of the third point.

Select a point value to view scoring criteria, solutions, and/or examples to score the response.

0	1	2	3

The student accurately includes all three of the criteria below.

- ☐ sets $u = 5 - x^2$
- ☐ integral in terms of u
- ☐ answer

Solution:

Let $u = g(x) = 5 - x^2$.

Then $du = -2x\ dx$ and $-\frac{1}{2}\ du = x\ dx$.

$x = 2 \Rightarrow u = 5 - 2^2 = 1$
$x = 1 \Rightarrow u = 5 - 1^2 = 4$

$$\int_1^2 x f'(g(x))\ dx = -\frac{1}{2}\int_4^1 f'(u)\ du = \frac{1}{2}\int_1^4 f'(u)\ du = \frac{1}{2}(-7) = -\frac{7}{2}$$